Mean-square stabilization for a class of networked multi-rate sampled-data systems

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***Abstract*—This work investigates the modelling and H_∞ control problem for a class of networked control systems with uncertainties and external disturbances, where the system state variables are divided and sampled by multiple groups of sensors with different sampling periods, i.e. multirate sampled-data systems. The multirate sampled data are transmitted over multiple networked channels with different network constraint parameters, such as different packet dropout rates and network-induced delays. Based on received asynchronous sampled data, a multirate state-feedback controller is introduced. To formulate the complex effects of the multirate sampling, network-induced delays, and statistic packet dropout, a new integral inequality is developed and utilized to construct a Lyapunov-like functional. Then the mean square stability criterion and control design method are given under a prescribed H_∞ performance level. Finally, the validity of the proposal result is testified by a vehicle active suspension system.**

***Index Terms*—Networked control systems, multirate sampling, statistic effects, integral inequalities.**

I. Introduction

Networked control systems (NCSs) have received a lot of attention in some field of theory and industrial applications [1]–[7] due to their advantages, such as saving wire, low cost, and easy installation and maintenance, where information among some devices is exchanged via networks. Different from a traditional control system, a networked control system often suffers from variant network constraints, such as networked-induced delays [8], and data packet dropout [9], etc. For simplicity, NCSs are often sampled by a single-rate sampling (SRS) way, i.e. all output variables of the system are sampled synchronously with a uniform sampling period. However, in many industrial applications, limited by sensor characteristics (such as their different spatial locations and/or types), the output variables of a system are often sampled by multiple sensors with different sampling rates (periods), which is called multirate sampling (MRS).

Multirate sampled-data systems (MR-SDSs) have attracted more and more interest in theory areas [10]–[16] and applications, such as vehicle systems [17], cyber-physical systems [18], and teleoperation systems [19]. With the rapid development and application of network technologies, networked MR-SDSs have recently received some attention [20]–[23]. [20] considered the stabilization problem linear networked MR-SDSs with logarithmic quantizers, where all sampling periods are proportional to a given base period. [21] investigated the multirate networked industrial process control problem in double-layer architecture, where the outputs of different layers are sampled by two different sampling periods. What's more, by introducing a matching mechanism, [22] designed a multirate observer-based controller for a class of networked MR-SDSs with bounded packet dropouts and bounded network-induced delays in a continuous-time framework, where it is not necessary to assume that all sampling periods are proportional. In general, [23] investigated the fuzzy dynamic output feedback control problem for a class of networked T-S fuzzy multirate sampled-data systems. However, stochastic factors are non-ignorable in some applications and they are not discussed under a continuous-time framework and the MRS case in these works above and reference therein. Thus, it is an interesting problem that how to design a multirate controller for an MR-SDS with multiple stochastic packet dropouts in a continuous-time framework.

In this paper, we focus on the modelling and H_∞ control for a class of networked MR-SDSs with stochastic packet dropouts, uncertainties, and external disturbances in a continuous-time framework. Using received multirate sampled-data, a multirate state-feedback controller is introduced. By input-delay approach [24] and the stochastic system approach [2], [25], the distribution characteristics of input-delays are considered.

The main contributions of this paper can be summarized as follows:

1) A more general relation is developed between two successfully transmitted sampling instants, while the randomness of packet dropouts isn't considered in [22], [23].

2) Considering the complexity of networked MR-SDSs in modeling, stability analysis, and controller design, a new integral inequality is developed to formulate the effects of the MRS, stochastic packet dropouts, and network-induced delays.

3) Using the integral inequality, a Lyapunov-like functional is constructed. Then the H_∞ performance analysis criterion and the controller design method are given for the networked

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MR-SDSs.

Notation. Let $\mathbb{N}$ (or $\mathbb{Z}^+$) is the set of non-negative integers (or positive integers). $\mathbb{R}^n$ denotes the n-dimensional Euclidian space with the norm $|z| = \sqrt{z^{\mathrm{T}}z}$, and $\mathbb{R}^{n\times m}$ stands for the set of $n \times m$ real matrices. I(or I_n) is an identity matrix with an appropriate dimension(or n-dimension). $\mathrm{sym(A)} = A+A^{\mathrm{T}}$. $diag\{\cdot\}$ denotes a block diagonal matrix. $col\{x, y\} = [x^{\mathrm{T}}, y^{\mathrm{T}}]^{\mathrm{T}}$, where x and y are column vectors. $prob\{.\}$ and $\mathbb{E}\{.\}$ stands for the probability and expectation of a stochastic variable, respectively.

II. Problem Statements

A. Plant

Consider the following plant

$$\mathcal{P}: \begin{cases} \dot{x}(t) &= Ax(t) + Bu(t) + B_f f(t,x) + B_\omega \omega(t) \\ z(t) &= Cx(t) + D_\omega \omega(t), t \geq 0, \end{cases} \tag{1}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^{n_u}$, $z(t) \in \mathbb{R}^{n_z}$, and $\omega(t) \in \mathbb{R}^{n_\omega}$ are the state, the measured output, the control signal, the controlled output, the external disturbance, and $\omega(t) \in \mathcal{L}_2[0,\infty)$, respectively. $f(t,x(t)) \in \mathbb{R}^l : [0,\infty) \times \mathcal{C}^1 \to \mathcal{L}_2[0,\infty)$ is a nonlinear function. A, B, B_f, B_ω, C, and D_ω are known constant matrices with appropriate dimensions. $B_f \in \mathbb{R}^{n\times l}$ is known constant matrix implying the structure information of nonlinearities and/or uncertainties. The nonlinear function $f(t,x(t))$ in (1) satisfies $f(t,0) = 0, \forall t \geq 0$, and the following quadratically inner-bounded condition

$$\begin{bmatrix} x(t) \\ f(t,x(t)) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} M_{11} & M_{12} \\ M_{12}^{\mathrm{T}} & M_{22} \end{bmatrix} \begin{bmatrix} x(t) \\ f(t,x(t)) \end{bmatrix} \geq 0 \tag{2}$$

where the symmetric matrices M_{11}, M_{22}, and matrix M_{12} are some known constant matrices with appropriate dimensions. The condition (2) was introduced in [26] and can describe some nonlinearities and/or uncertainties, such as norm-bounded parameter uncertainties [27], Lipschitz conditions, and saturation nonlinearities.

B. Some preliminaries for networked multirate sampled-data systems

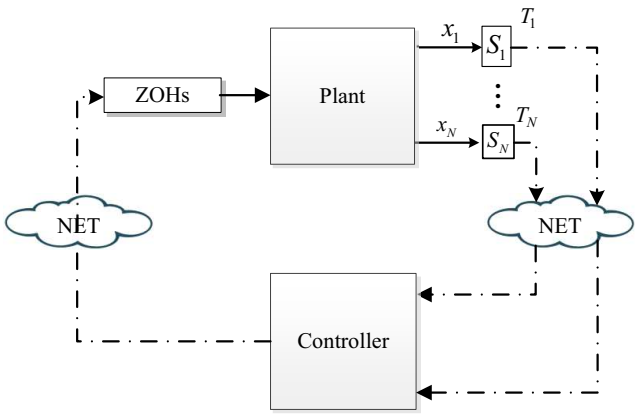


Fig. 1. The schematic diagram of networked MR-SDSs

The objective of this paper is to investigate the H_∞ control problem for the plant (1) with the MRS, and its schematic diagram is shown in Figure 1.

In Figure 1, the state $x(t)$ of the plant (1) is sampled by the MRS. Specifically, The plant (1) has m state variables, and they are classified into N groups according to physical characteristics of sensors, i.e., $x(t) = \begin{bmatrix} x_1^{\mathrm{T}}(t) & \cdots & x_N^{\mathrm{T}}(t) \end{bmatrix}^{\mathrm{T}}$. Then the i-th sub-vector $x_i(t) \in \mathbb{R}^{n_i}$ is sampled by the sensor group S_i with a uniform sampling period T_i, and the different sensor groups can have different sampling periods, i.e., $T_i \neq T_j$ for $i \neq j$ and $i, j \in \mathbb{I}_N := \{1, 2, ..., N\}$.

Furthermore, the sampling time sequence of S_i is denoted by the set $\{kT_i | k \in \mathbb{N}\}$, $i \in \mathbb{I}_N$. The sampled data of different output sub-vectors are individually sent to the filter through different communication channels with stochastic packet dropouts. The controller and the actuator are event-driven. The controller updates its output based on the latest received sampled-data of every state component when a sampled-data arrives. s_i^k ($i \in \mathbb{I}_N$) are the k-th successfully transmitted sampling time-stamp to the zero-order-holders (ZOHs) of the i-th channel, $k \in \mathbb{N}$. $t_i^k := s_i^k T_i + \tau_i$ is the first update time instant of the ZOHs to $x_i(s_i^k T_i)$. The delays τ_i include mainly the network transmission delays of sensor-to-controller and controller-to-ZOHs, and some waiting time of ZOHs.

C. Multirate controller

In this subsection, the following multirate state-feedback controller is considered for the networked MR-SDSs

$$u(t) = \sum_{i=1}^{N} F_i x_i(t - \tau_i(t)) \tag{3}$$

where $F_i (i \in \mathbb{I}_N)$ are the controller gains to be determined and the input-delays $\tau_i(t) = t - s_i^k T_i$ over $t \in [t_i^k, t_i^{k+1})$ for $k \in \mathbb{N}$ and $i \in \mathbb{I}_N$.

Combining (1) and (3), the resultant closed-loop system can be described as

$$\begin{cases} \dot{x}(t) &= \bar{A}x(t) + \bar{B}e(t) + B_f f(x) + B_\omega \omega(t) \\ z(t) &= Cx(t) + D_\omega \omega(t), t \geq t_0, \end{cases} \tag{4}$$

where $e(t) = col\{e_i(t)\}_{i\in\mathbb{I}_N}$, $e_i(t) = x_i(t-\tau_i(t)) - x_i(t)$, $\bar{A} = A + BF$, $F = \begin{bmatrix} F_1 & \cdots & F_N \end{bmatrix}$ and $\bar{B} = BF$. What's more, define the initial instant $t_0 = \max_{j\in\mathbb{I}_N}\{t_j^0\}$ and the initial condition $x(t) = \phi(t), t \in [0, t_0)$ of the system (4).

Note that the upper bounds of the input delays $\tau_i(t)$ ($i \in \mathbb{I}_N$) from the MRS may be large due to excessive successive packet dropouts, while they may be small at most time. Inspired by the stochastic system approach in [2], we consider their distribution characteristics.

D. Distributions of the input delays

For the i-th ($i \in \mathbb{I}_N$) channel of the system, α_i^k is a Bernoulli distribution representing the occurrence of packet dropouts. Also, $prob\{\alpha_i^k = 1\} = \rho_i$ and $prob\{\alpha_i^k = 0\} = 1 - \rho_i$, $k \in \mathbb{N}$.

Denote $d_i^k := s_i^{k+1} - s_i^k = \alpha_i^{s_i^k} + \alpha_i^{s_i^k+1} \cdots + \alpha_i^{s_i^{k+1}}$. Then d_i^k is a stochastic variable with

$$prob\{d_i^k = j\} = \begin{cases} \rho_i^{j-1}(1-\rho_i), & j = 1, ..., d_i - 1, \\ \rho_i^{d_i-1}, & j = d_i. \end{cases}$$

where ρ_i is the package dropout rate from the i-th sensor group to ZOHs and $d_i \in \mathbb{N}$ is its maximum allowable bound of successive packet dropouts, and $i \in \mathbb{I}_N$.

Define some sets $\Omega_{ij} = \{t|\tau_i(t) \in [(j-1)T_i + \tau_i, jT_i + \tau_i)\}$ satisfying $\cup\Omega_{ij} = [t_0, \infty)$, where $i \in \mathbb{I}_N$, $j \in \mathbb{I}_{d_i}$, $k \in \mathbb{N}$. Also, define some auxiliary delays $\tau_i^0(t) = t - kT_i$, $t \in [kT_i + \tau_i, (k+1)T_i + \tau_i)$, $i \in \mathbb{I}_N$, and $k \in \mathbb{N}$. Furthermore, we introduce the following stochastic variables

$$\beta_{ij}(t) = \begin{cases} 1, & t \in \Omega_{ij}, \\ 0, & t \notin \Omega_{ij}, \end{cases}$$

and some auxiliary delays

$$\tau_{ij}(t) = \begin{cases} \tau_j(t), & t \in \Omega_{ij}, \\ \tau_j^0(t), & t \notin \Omega_{ij}. \end{cases}$$

Then, it is not hard to find that $\beta_{ij}(t) (i \in \mathbb{I}_N$ and $j \in \mathbb{I}_{d_i})$ are some Bernoulli distributed sequence satisfying $\sum_{j=1}^{d_i} \beta_{ij}(t) = 1$, $\mathbb{E}\{\beta_{ij}(t)\beta_{ik}(t)\} = 0$ for $j \neq k$ and $\mathbb{E}\{\beta_{ij}^2(t)\} = \mathbb{E}\{\beta_{ij}(t)\} = prob\{t \in \Omega_{ij}\} = \theta_{ij}$, where $\theta_{ij} = \frac{1-\rho_i}{\rho_i}\int_0^{\rho_i} \frac{\theta^{j-1} - \theta^{d_i-1})}{1-\theta} d\theta + \frac{\rho_i^{d_i-1}}{d_i}$ and $\int_0^{\rho_i} \frac{\theta^{j-1}}{1-\theta} d\theta = \begin{cases} -\ln(1-\rho_i), & j = 1, \\ -\ln(1-\rho_i) - \rho_i - ... - \frac{(\rho_i)^{j-1}}{j-1}, & j > 1. \end{cases}$

Furthermore, the system (5) can be rewritten as the following stochastic model

$$\begin{cases} \dot{x}(t) &= \bar{A}x(t) + \bar{B}e(t) + B_f f(x) + B_\omega \omega(t) \\ z(t) &= Cx(t) + D_\omega \omega(t), t \geq t_0, \end{cases} \quad (5)$$

where $e(t) = col\{e_i(t)\}_{i\in\mathbb{I}_N}$, $e_i(t) = x_i(t - \tau_i(t)) - x_i(t)$, and $x_i(t - \tau_i(t)) = \sum_{j=1}^{d_i} \beta_{ij}(t)(x_i(t - \tau_{ij}(t))$.

Remark 1: Inspired from the stochastic system approach [2], [25], the system (4) is equivalently transformed into the stochastic system (5), where the input-delay $\tau_i(t)$ with larger upper-bounds is sliced as d_i input-delays $\tau_{ij}(t) (j \in \mathbb{I}_{d_i})$ with smaller upper-bounds by considering the distribution characteristic of $\tau_i(t)$. Less conservative results may be obtained than those without considering its distribution characteristic.

Next, this paper focuses on designing the multirate controller (3) for the system (1), such that the resultant closed-loop system (5) is mean-square asymptotically stable with a prescribed H_∞ performance level γ defined as follows.

- When $\omega(t) \equiv 0$, the system (5) is asymptotically stable in a mean-square sense, i.e., $\lim_{t\to\infty} \mathbb{E}\{|x(t)|^2\} = 0$ for any initial condition;
- Under zero initial condition, i.e., $\phi(t) \equiv 0$, $t \in [0, t_0)$, the inequality

$$\mathbb{E}\{\int_{t_0}^{\infty} z^{\mathrm{T}}(s)z(s)ds\} \leq \gamma^2 \int_{t_0}^{\infty} \omega^{\mathrm{T}}(s)\omega(s)ds$$

holds for all nonzero $\omega(t) \in \mathcal{L}_2[t_0, \infty)$, and a prescribed scalar $\gamma > 0$.

The following lemmas are given and utilized to facilitate theoretical analysis.

Lemma 1 ([28]): For $X > 0$, if $\tau(t)$ is an almost everywhere differentiable function satisfying $0 \leq \tau(t) \leq \bar{\tau}$ and $0 \leq \dot{\tau}(t) \leq 1$ for $t \in [0, \infty)$, then the following inequality holds:

$$\int_0^{\infty} (x(s) - x(s - \tau(s)))^{\mathrm{T}} X (x(s) - x(s - \tau(s)))ds \leq \bar{\tau}^2 \int_0^{\infty} \dot{x}^{\mathrm{T}}(s)^{\mathrm{T}} X \dot{x}(s)ds \quad (6)$$

where $\dot{x}(t) \in \mathcal{L}_2[0, \infty)$ and $\dot{x}|_{[-\infty,0]} \equiv 0$.

Lemma 2 (Barbalat lemma, [29], [30]): If $x(t), \dot{x}(t)$ are bounded and $x(t)$ is absolutely integrable, then $\lim_{t\to\infty} x(t) = 0$.

III. Main results

It is noted that $e(t)$ in (5) contains Nd_i input-delays and excessive input-delays can sharply increase the complexity in stability analysis and controller design. To this end, a new integral inequality is developed to describe $e(t)$ overall. Using the proposal inequality, a new Lyapunov-like functional is developed. Then the H_∞ performance analysis criterion and the controller design method are given for the networked MR-SDSs (5), respectively.

A. An integral inequality

In this subsection, an integral inequality is given to formulate the perturbed term $e(t)$ in (5).

Theorem 1: For $t \geq t_0$ and $X \in S_+^n$, the following inequality holds:

$$\int_{t_0}^{t} \mathbb{E}\{e^{\mathrm{T}}(s)Xe(s)\}ds \leq \int_0^t \mathbb{E}\{\dot{x}^{\mathrm{T}}(s)D^{\mathrm{T}}XD\dot{x}(s)\}ds \quad (7)$$

where $X = diag\{X_i\}_{i\in\mathbb{I}_N}, X_i \in \mathbb{S}_+^{n_i}, D = diag\{\delta_i I_{n_i}\}_{i\in\mathbb{I}_N}$, $\delta_i = \sqrt{\mathbb{E}\{(d_i^k)^2\}T_i^2 + 2\mathbb{E}\{d_i^k\}T_i\tau_i + \tau_i^2}$, $\mathbb{E}\{d_i^k\} = \frac{1}{2}(1 + \frac{1-(d_i+1)\rho_i^{d_i-1}+d_i\rho_i^{d_i}}{1-\rho_i} + \frac{(d_i+1)\rho_i^{d_i+1}}{2})$, and $\mathbb{E}\{(d_i^k)^2\} = \frac{6-3\rho_i+\rho_i^2-(4d_i+5)\rho_i^{d_i}+4d_i\rho_i^{d_i+1}}{6(1-\rho_i)^2}$ for $i \in \mathbb{I}_N$.

Proof: Following (5), it yields

$$e_i(t) = \sum_{j=1}^{d_i} \beta_{ij}(t)(x_i(t - \tau_{ij}(t)) - x_i(t))$$

and

$$\int_{t_0}^{t} \mathbb{E}\{e_i^{\mathrm{T}}(s)X_i e_i(s)\}ds = \int_0^t \sum_{j=1}^{d_i} \mathbb{E}\{\beta_{ij}^2(s)\}e_{ij}^{\mathrm{T}}(s)X_i e_{ij}(s)ds = \sum_{j=1}^{d_i} \theta_{ij} \int_0^t \mathbb{E}\{e_{ij}^{\mathrm{T}}(s)X_i e_{ij}(s)\}ds,$$

where $e_{ij}(t) = x_i(t - \tau_{ij}(t)) - x_i(t)$.

Notice that $0 \leq \tau_{ij}(t) \leq jT_i + \tau_i$ and $0 \leq \dot{\tau}_{ij}(t) \leq 1$. Construct a function $\tilde{x}(t)$ over the interval $[0, \infty)$, where

$\dot{\tilde{x}}(t) = \dot{x}(t)$ for $t \in [t_0, t]$, otherwise $\dot{\tilde{x}}(t) = 0$. Applying Lemma 1 for $\tilde{x}(t)$, we have

$$\int_{t_i^0}^{t} \mathbb{E}\{e_{ij}^{\mathrm{T}}(s) X_i e_{ij}(s)\} ds = \mathbb{E}\{\int_{t_i^0}^{t} e_{ij}^{\mathrm{T}}(s) X_i e_{ij}(s) ds\} \leq (jT_i + \tau_i)^2 \int_0^t \mathbb{E}\{\dot{x}_i^{\mathrm{T}}(s) X_i \dot{x}_i(s)\} ds.$$

Then

$$\int_{t_i}^{t} \mathbb{E}\{e_i^{\mathrm{T}}(s) X_i e_i(s)\} ds \leq \delta_i^2 \int_0^t \mathbb{E}\{\dot{x}_i^{\mathrm{T}}(s) X_i \dot{x}_i(s)\} ds$$

where $\delta_i = \sqrt{\mathbb{E}\{(d_i^k)^2\} T_i^2 + 2\mathbb{E}\{d_i^k\} T_i \tau_i + \tau_i^2}$.
Thus, $\int_{t_0}^{t} \mathbb{E}\{e^{\mathrm{T}}(s) X e(s)\} ds \leq \int_0^t \mathbb{E}\{\dot{x}^{\mathrm{T}}(s) D^{\mathrm{T}} X D \dot{x}(s)\} ds$. ■

B. H_∞ performance analysis

Inspired from the integral inequality (7) in Theorem 1, the Lyapunov-like functional $V(t)$ is constructed as follows:

$$V(t) = x^{\mathrm{T}}(t) P x(t) + \int_0^t \mathbb{E}\{\dot{x}^{\mathrm{T}}(s) D^{\mathrm{T}} X D \dot{x}(s)\} ds - \int_{t_0}^{t} \mathbb{E}\{e^{\mathrm{T}}(s) X e(s)\} ds,\ t \geq t_0, \tag{8}$$

where $P > 0$ and $X > 0$ (defined in Theorem 1).

Based on the Lyapunov-like functional, Theorem 2 guarantees that the system (5) is mean-square asymptotically stable with a prescribed H_∞ performance level γ.

Theorem 2: For given positive scalars ρ and γ, matrices D and F, if there exist matrices $P > 0$, $X > 0$, M_1, and M_2 such that the following linear matrix inequality holds

$$\begin{bmatrix} \Pi_{11} & \Pi_{12} & M_1^{\mathrm{T}}\bar{B} & M_1^{\mathrm{T}}B_\omega & {M_1}^{\mathrm{T}}B_f & 0 \\ * & \Pi_{22} & M_2^{\mathrm{T}}\bar{B} & M_2^{\mathrm{T}}B_\omega & M_2^{\mathrm{T}}B_f + \rho M_{12} & C^{\mathrm{T}} \\ * & * & -X & 0 & 0 & 0 \\ * & * & * & -\gamma^2 I & 0 & D_\omega^{\mathrm{T}} \\ * & * & * & * & \rho M_{22} & 0 \\ * & * & * & * & * & -I \end{bmatrix} < 0 \tag{9}$$

where D and X are defined in Theorem 1, $\Pi_{11} = -\mathrm{sym}\{M_1\} + D^{\mathrm{T}} X D$, $\Pi_{12} = P - M_2 + M_1^{\mathrm{T}}\bar{A}$, $\Pi_{22} = \mathrm{sym}\{M_2^{\mathrm{T}}\bar{A}\} + \rho M_{11}$. Then the system (5) is mean-square asymptotically stable with a prescribed H_∞ performance level γ.

Proof: Following [2], define an operator with respect to $V(t)$ as follows:

$$\mathfrak{L}\{V(t)\} := \lim_{\triangle \to 0^+} \frac{1}{\triangle}\{\mathbb{E}\{V(t+\triangle)|t\} - V(t)\}. \tag{10}$$

Compute $\mathfrak{L}\{V(t)\}$ along the of (5),

$$\mathfrak{L}\{V(t)\} = \mathbb{E}\{\dot{x}^{\mathrm{T}}(t) P x(t) + x^{\mathrm{T}}(t) P \dot{x}(t) + \dot{x}^{\mathrm{T}}(t) D^{\mathrm{T}} X D \dot{x}(t) - e^{\mathrm{T}}(t) X e(t)\}. \tag{11}$$

Based on (5), it yields

$$0 = \mathbb{E}\{2\varsigma_0^{\mathrm{T}}(t)(e_1^{\mathrm{T}} M_1^{\mathrm{T}} + e_2^{\mathrm{T}} M_2^{\mathrm{T}}) \times [\bar{A} e_2 + \bar{B} e_3 + B_\omega e_4 + B_f e_5 - e_1]\varsigma_0(t)\}, \tag{12}$$

where M_1 and M_2 are slack matrices to be determined, $\varsigma_0(t) := col\{\dot{x}(t), x(t), e(t), \omega(t), f(t)\}$, and $\varsigma_0(t)\ e_i$, $i \in V_5$, such as $x(t) = e_2 \varsigma_0(t)$.

Combining (2), (9), (11) and (12), there exists a sufficient small $\varepsilon > 0$ such that

$$\mathbb{E}\{\mathfrak{L}\{V(t)\} + z^{\mathrm{T}}(t) z(t) - \gamma^2 \omega^{\mathrm{T}}(t)\omega(t)\} \leq \mathbb{E}\{\varsigma_0^{\mathrm{T}}(t)\Omega\varsigma_0(t)\} < -\varepsilon \mathbb{E}\{|x(t)|^2\} < 0. \tag{13}$$

Integrating the two sides of (13) from t_0 to t, it yields

$$\mathbb{E}\{V(t) - V(t_0) + \int_{t_0}^{t} z^{\mathrm{T}}(s) z(s) ds\} - \gamma^2 \int_{t_0}^{t} \omega^{\mathrm{T}}(s)\omega(s) ds \leq -\varepsilon \int_{t_0}^{t} \mathbb{E}\{|x(s)|^2\} ds \leq 0. \tag{14}$$

Under zero initial condition and $V(t) > 0$, one can obtain $\mathbb{E}\{\int_{t_0}^{\infty} z^{\mathrm{T}}(s) z(s) ds\} \leq \gamma^2 \int_{t_0}^{\infty} \omega^{\mathrm{T}}(s)\omega(s) ds$.

When $\omega(t) \equiv 0$, we have $\mathbb{E}\{V(t) - V(t_0)\} < -\varepsilon \int_{t_0}^{t} \mathbb{E}\{|x(s)|^2\} ds < 0$. Since $0 \leq \mathbb{E}\{x^{\mathrm{T}}(t) P x(t)\} \leq \mathbb{E}\{V(t_0)\}$, then $\int_{t_0}^{\infty} \mathbb{E}\{|x(s)|^2\} ds < \infty$; that's to say, $\varphi(t) := \mathbb{E}\{|x(t)|^2\}$ is bounded and absolutely integrable. Then $\mathbb{E}\{|e(t)|^2\}$is also bounded. What's more, since $\mathbb{E}\{\mathfrak{L}\{V(t)\}\} < 0$ and $\dot{x}^{\mathrm{T}}(t) P x(t) + x^{\mathrm{T}}(t) P \dot{x}(t) \leq \rho x^{\mathrm{T}}(t) P x(t) + \frac{1}{\rho}\dot{x}^{\mathrm{T}}(t) P \dot{x}(t)$ for arbitrary $\rho > 0$, one can get $\mathbb{E}\{\dot{x}^{\mathrm{T}}(t)(D^{\mathrm{T}} X D - \frac{1}{\rho} P)\dot{x}(t)\} \leq \mathbb{E}\{e^{\mathrm{T}}(t) X e(t) + \rho x^{\mathrm{T}}(t) P x(t)\}$. By choosing a sufficiently large positive scalar ρ such that $D^{\mathrm{T}} X D - \frac{1}{\rho} P > 0$, we can show that $\mathbb{E}\{|\dot{x}(t)|^2\}$ is bounded on the interval $[t_0, \infty)$. Applying Cauchy-Schwarz inequality, $\dot{\varphi}(t) = \mathbb{E}\{2\dot{x}^{\mathrm{T}}(t) x(t)\}^2 \leq 4\mathbb{E}\{|\dot{x}(t)|^2\}\mathbb{E}\{|x(t)|^2$, i.e $\dot{\varphi}(t)$ is bounded on the interval $[t_0, \infty)$. Using Lemma 2, then $\lim_{t\to\infty} \varphi(t) = 0$, i.e $\lim_{t\to\infty} \mathbb{E}\{|x(t)|^2\} = 0$.

Therefore, the system (5) is mean-square asymptotically stable with a prescribed H_∞ performance level γ. ■

C. Multirate controller design

The design method of the multirate controller (3) is given by Theorem 3.

Theorem 3: For given scalars α, β, $\gamma > 0$, and a matrix D (defined in Theorem 1), if there exists a scalar $\rho > 0$, matrices $\widehat{P} > 0$, $\hat{X} > 0$, W, and $\hat{F}$, such that the following inequality holds:

$$\begin{bmatrix} \tilde{\Pi} & \widetilde{W}^{\mathrm{T}} \\ * & -\widetilde{X} \end{bmatrix} < 0 \tag{15}$$

where $\tilde{\Pi} = \begin{bmatrix} -\mathrm{sym}\{W\} & \tilde{\Pi}_{12} & B\hat{F} & B_\omega & \tilde{\rho} B_f & 0 \\ * & \tilde{\Pi}_{22} & \alpha B\hat{F} & \alpha B_\omega & \tilde{\Pi}_{25} & W^{\mathrm{T}} C^{\mathrm{T}} \\ * & * & \tilde{\Pi}_{33} & 0 & 0 & 0 \\ * & * & * & -\gamma^2 I & 0 & D_\omega^{\mathrm{T}} \\ * & * & * & * & \rho M_{22} & 0 \\ * & * & * & * & * & -I \end{bmatrix}$,
$\widetilde{W} = \begin{bmatrix} 0 & \sqrt{M_{11}} W & 0 & 0 & 0 & 0 \\ DW & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ and $\widetilde{X} = diag\{\rho I, \hat{X}\}$, with$\tilde{\Pi}_{12} = \widehat{P} - \alpha W^{\mathrm{T}} + AW + B\hat{F}$, $\tilde{\Pi}_{22} = \alpha\,\mathrm{sym}\{AW + B\hat{F}\}$, $\tilde{\Pi}_{25} = \alpha B_f + \rho W^{\mathrm{T}} M_{12}$, $\tilde{\Pi}_{33} = \beta^2 \hat{X} - \beta\mathrm{sym}\{W\}$. Then the system (5) is mean square asymptotically stable

with a prescribed H_∞ performance level γ. Furthermore, the controller gain is given by $F = \hat{F}W^{-1}$.

Proof: This theorem will be proved based on Theorem 2.

Take $M_1^{-1} = W$ and $M_1 = \alpha M_2$, where $\alpha > 0$. Let $H = diag\{W, W, W, I, I\}$ $\hat{X} = X^{-1}$, $\widehat{P} = W^{\mathrm{T}}PW$ and $\hat{F} = FW$. Since $(W + \beta\hat{X})^{\mathrm{T}}\hat{X}^{-1}(W + \beta\hat{X}) \geq 0$ for a given scalar β, $-W^{\mathrm{T}}\hat{X}^{-1}W \leq \beta^2\hat{X} - \beta\ \mathrm{sym}\{W\}$. Thus, using Schur complement the inequality (15) holds. ■

IV. Illustrative examples

Here, we consider the following vehicle active suspension system ([17]) with the form (1), where $x(t) = col\{x_1(t), x_2(t), x_3(t), x_4(t)\}$,

$$A = \begin{bmatrix} 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \\ -\frac{k_s}{m_s} & 0 & -\frac{c_s}{m_s} & \frac{c_s}{m_s} \\ \frac{k_s}{m_u} & -\frac{k_t}{m_u} & \frac{c_s}{m_u} & -\frac{c_s+c_t}{m_s} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{m_s} \\ -\frac{1}{m_u} \end{bmatrix},$$

$$B_\omega = \begin{bmatrix} 0 & -1 & 0 & \frac{c_t}{m_u} \end{bmatrix}^{\mathrm{T}}, \quad C = \begin{bmatrix} -\frac{k_s}{m_s} & 0 & -\frac{c_s}{m_s} & \frac{c_s}{m_s} \end{bmatrix},$$

$$B_f = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}^{\mathrm{T}}, \quad D_\omega = \frac{1}{m_s},$$

and $m_s = 973\mathrm{kg}$, $m_u = 114\mathrm{kg}$, $k_s = 42720\mathrm{N/m}$, $k_t = 101115\mathrm{N/m}$, $c_s = 1095\mathrm{Ns/m}$, $c_t = 14.6\mathrm{Ns/m}$; $x_1(t)$ is the suspension deflection, $x_2(t)$ is the tire deflection, $x_3(t)$ is the sprung mass speed and $x_4(t)$ is the unsprung mass speed, respectively. $f(x)$ is an uncertain term, and the road curvature is viewed as an external disturbance $\omega(t)$, which is depicted in Figure 2.

Due to the different types of the used sensors, the former two state variables $x_1(t)$ and $x_2(t)$ are sampled by the sensor group S_1 with the sampling period T_1 and the latter two state variables $x_3(t)$ and $x_4(t)$ are sampled by the sensor group S_2 with the sampling period T_2, where $T_2 > T_1$. As shown in [22], [31], the corresponding single-rate sampling period has to be selected as the maximum of all sampling periods, i.e. $T = \max\{T_1, T_2\}$.

In the sequel, we will compare the MRS method with the SRS method and shows that our MRS method can some better H_∞ performance levels under different sampling periods.

Next, by taking $\alpha = 1, \beta = 100$, $\rho_1 = \rho_2 = 20\%$, $d_1 = d_2 = 100$, $M_{11} = diag\{0, 0, 0, 0.01\}, M_{12} = 0$, and $M_{11} = -1$. Then set $T_2 = 2T_1$ and using Theorem 2, Figure 3 shows the minimal achievable performance γ_{min} of the MRS method and those of the SRS method as the sampling period T increases, where $T = \max\{T_1, 2T_1\}$. It shows that the MRS method can obtain better H_∞ performance levels than the SRS method.

Furthermore, taking $T_1 = 40ms$ and $T_2 = 120ms$, the controller gains can be obtained as follows: $F_1 = 10^4 \times \begin{bmatrix} -0.002 & -0.001 \\ 2.1867 & 1.2969 \end{bmatrix}$ and $F_2 = 10^4 \times \begin{bmatrix} -0.004 & -0.001 \\ 3.7533 & 1.2372 \end{bmatrix}$. Also, the initial state of the system (5) is given by $\begin{bmatrix} 0.2 & -0.1 & 0 & 0 \end{bmatrix}^{\mathrm{T}}$, respectively. Then Figure 4 gives the state trajectories of the vehicle active suspension system with a multirate controller (3) under package dropout rates $\rho_1 = 20\%$ and $\rho_2 = 20\%$. It can be seen that the state trajectories of the vehicle active suspension system converge to the equilibrium point after two curved roads.

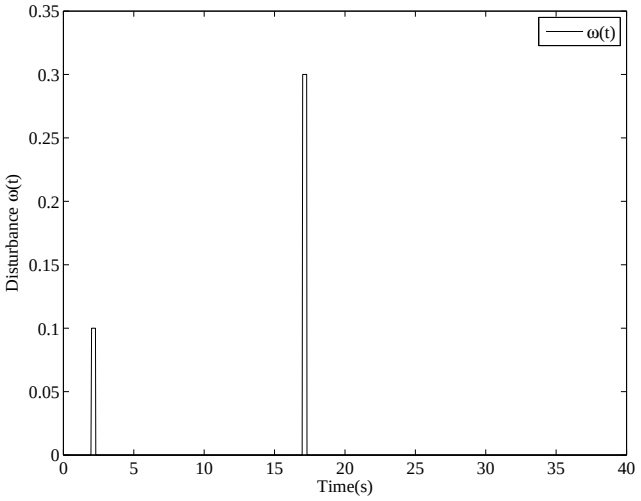


Fig. 2. The external disturbance from a curved road

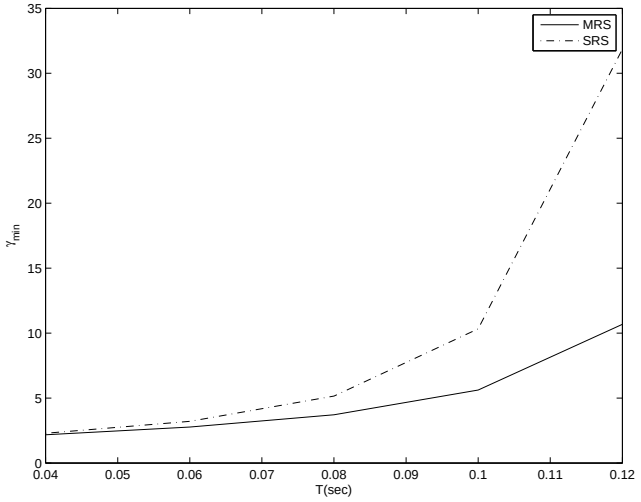


Fig. 3. Minimal achievable performance γ_{min} of the MRS method and those of the SRS method

V. Conclusion

This paper has investigated the modelling and H_∞ control for a class of nonlinear networked MR-SDSs with stochastic packet dropouts. The effects of the MRS and multiple stochastic network channels have been formulated by an integral inequality. Using the inequality, a Lyapunov-like functional has been constructed. Then the H_∞ performance analysis criterion and the control design method have been given in terms of linear matrix inequities. At last, compared with an SRS method, a vehicle active suspension system has shown the effectiveness and advantages of the proposed MRS method. One of the future research would be to investigate networked

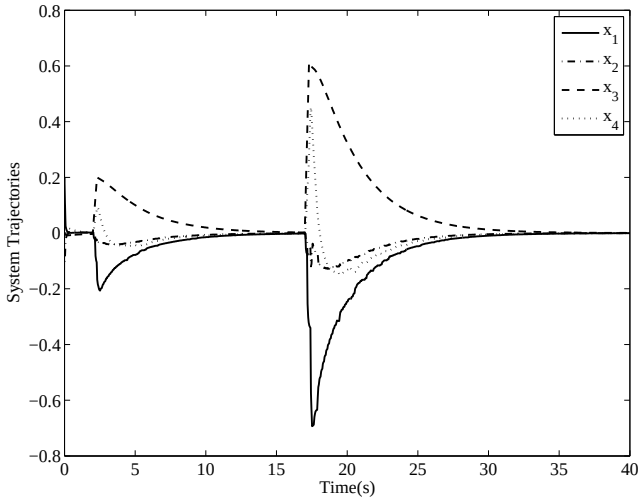


Fig. 4. State trajectories of the vehicle active suspension system with $\rho_1 = 20\%$ and $\rho_2 = 20\%$

control for a more general nonlinear system with stochastic network channels in a continuous-time framework.

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